AIAA JOURNAL
Vol. 46, No. 2, February 2008

Solution to the Incorrect Benchmark Shell-Buckling Problem

Brian L. Wardle*
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

DOI: 10.2514/1.26698

A benchmark, geometrically nonlinear, shell-buckling problem is reviewed and the correct solution is identified and discussed vis-à-vis the incorrect solution from the open literature. The problem is that of a hinged, thin, isotropic cylindrical shell section, point-loaded transversely at the center, undergoing large deformations (5× thickness). This benchmark problem and solution were introduced in 1972 and have been reproduced in at least 30 journal articles and books in the more than three decades since that time. The benchmark problem is of interest because it demonstrates many features possible in shell buckling, including a load-limit point (snap-through buckling) and a deflection-limit point (snap-down, or snap-back). The benchmark problem is typically used to demonstrate the capability of commercial/private finite element codes to traverse such complicated nonlinear load paths. The existing incorrect benchmark solution involves the nonlinear growth of only symmetric deformation modes. The correct solution to this benchmark problem involves bifurcation from the nonlinear load path into a simple asymmetric deformation mode. Experimental data do not exist for verification of the calculated benchmark response, and so a similar problem using a laminated composite shell is suggested as an alternative benchmark problem for shell stability analysis.

Nomenclature

E	=	isotropic Young's modulus
E_{11}	=	composite-ply modulus in the fiber direction
E_{22}	=	composite-ply modulus transverse to the fiber direction
G_{12}	=	composite-ply shear modulus
h	=	height of the shell panel
L	=	side length of the shell panel
R	=	radius of the shell panel
Ru, Rv, Rw	=	rotations about the curvilinear coordinate axes
S	=	span of the shell panel
t	=	thickness of the shell panel
x, y, z	=	curvilinear coordinates
u, v, w	=	displacements corresponding to curvilinear coordinates
w_c	=	vertical deflection of the center of the shell panel
β	=	half the included angle of the benchmark circular cylindrical shell panel
ν	=	isotropic Poisson's ratio
ν_{12}	=	composite-ply major Poisson's ratio

I. Introduction

A BENCHMARK, geometrically nonlinear, shell-buckling problem from the literature is reviewed and the correct solution identified and discussed vis-à-vis the incorrect solution. This benchmark problem and solution were introduced in 1972 by Sabir and Lock [1] and have appeared in numerous journal articles and books in more than 30 years since that time (sample references from each of the intervening decades are in [2–7]). The benchmark problem has been used over the years to investigate advances in numerical [especially, finite element (FE)] methods for handling load and/or deflection reversals in nonlinear buckling problems. It is also used to demonstrate the capability of FE codes to traverse such complicated load paths and appears as a validation problem for shell modeling in FE manuals (e.g., [8]†). The benchmark problem is an isotropic, thin, circular cylindrical shell panel of square planform, transversely point-loaded, undergoing large deformations (5× thickness at the center), including buckling and postbuckling.

The long-standing benchmark solutions in the literature show a complex limit-point response; the benchmark solution for this shell involves both load- and deflection-limit points. At these limit points, the shell response contains snap-through and snap-down/snap-back, respectively (see the discussion of both for a system of pinned bars in [9]). The correct solution to this problem involves bifurcation before the snap-through limit point into an asymmetric deformation mode. Although the benchmark problem is of a shell panel, the overall response is analogous to a centrally point-loaded arch, illustrated in Fig. 1. Such arches either buckle at a load-limit point into a spanwise symmetric deformation mode or bifurcate into a spanwise asymmetric deformation mode. The asymmetric bifurcation mode is well known, as discussed by Timoshenko and Gere [10] with the asymmetric mode, oftentimes termed "inextensional" or "extensionless," known through the work of Lord Rayleigh and applied to the asymmetric deformation of arches by Hurlbrink (see Timoshenko and Gere [10] for early references). Bifurcation vs limit-point response for the arch is usually associated with the ratio of the arch midspan height h (or rise) to the span S for an isotropic arch, with bifurcation more likely for higher ratios [11] (deeper than shallow arches); the parameter governing the response involves both material and geometric quantities in the case of an orthotropic arch [12]. Thus, the behavior of arches indicates that a configuration such as the benchmark shell may either bifurcate or buckle at a load-limit point; the benchmark problem is shown here to behave like a deep arch and bifurcate before reaching the first limit point.

No experimental data exist for verification of the calculated benchmark response, and so a similar problem for a composite shell is suggested as an alternative benchmark problem for shell stability analysis, because both load-deflection and detailed mode-shape data are available and in excellent agreement with FE predictions.

II. Benchmark Problem and Solutions

The benchmark problem is first detailed, along with a summary of the FE analysis approach (a full description [13] of the FE procedures

Presented as Paper 2028 at the 47th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Newport, RI, 1–4 May 2006; received 21 July 2006; revision received 2 October 2007; accepted for publication 19 October 2007.

*Assistant Professor, Department of Aeronautics and Astronautics, Building 33-314, 77 Massachusetts Avenue. Member AIAA.

†ABAQUS version 6.7 shows the same solution as that shown in version 6.6 [8].

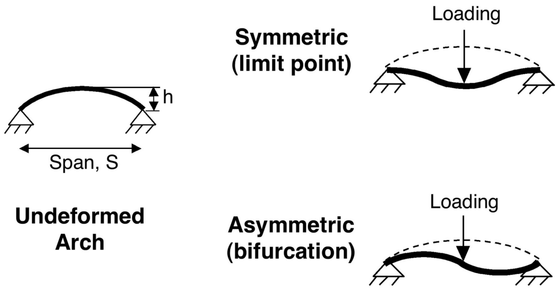

Fig. 1 Limit-point and bifurcation response for an arch showing symmetric and asymmetric spanwise mode shape; *h* is the arch height at midspan.

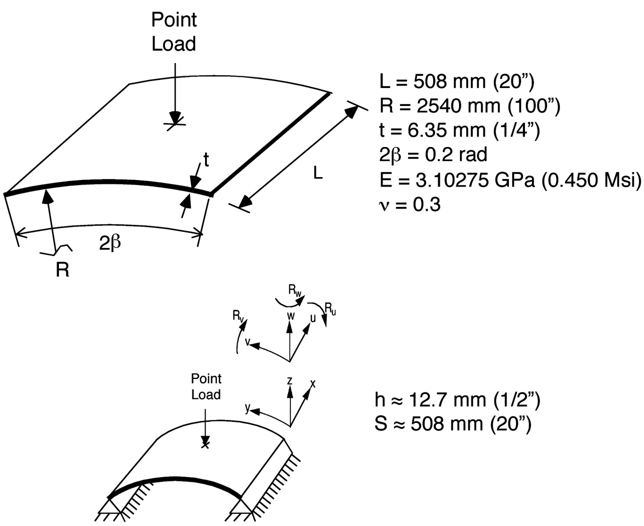

Fig. 2 Geometric and material data for benchmark shell problem, and loading, boundary conditions, coordinates, and displacement/rotation conventions.

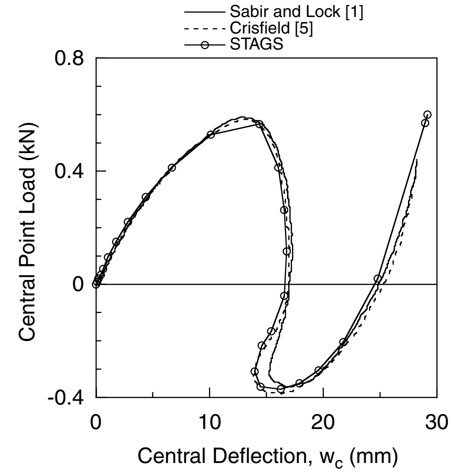

Fig. 3 Load-deflection response solutions (incorrect-symmetric) to the benchmark problem from the literature and using the STAGS FE code.

is available). The incorrect solution to the benchmark problem is first presented and discussed so that it can be compared with the correct solution. For the purpose of clarity, the solution to the benchmark that corresponds to that found in the literature to date will be termed the "incorrect-symmetric" solution and involves only spanwise symmetric deformations. The correct solution will be termed "correct-bifurcation," and it is understood that the shell bifurcates into a spanwise asymmetric mode, as in Fig. 1.

A. Problem Geometry and Analysis Summary

Material and geometric properties for the benchmark problem are given in Fig. 2. The panel spans and is supported by two straight axial edges having a pinned fixture that cannot move (i.e., hinged). The panel is free of traction on the curved circumferential edges that have a midspan height h. The hinged constraint is represented in the model by setting all nodal displacements and rotations ($u = v = w = Rv = Rw$) equal to zero, except rotation about the x axis (Ru), where zero moment is enforced. In the analysis, the shell is transversely point-loaded at the center of the shell surface. Quarter-symmetry was not assumed because a full model of the shell is required to fully investigate bifurcation.

The structural analysis of general shells (STAGS) [14] FE code was used to obtain the nonlinear shell response. The 410 shell element available in STAGS is used for its applicability to the thin shell structures in this work [15]. This commonly used element employs the nonlinear Kirchhoff–Love shell hypothesis, which ignores transverse shear, but is known to be extremely accurate for modeling thin shell structures. It is a displacement-based four-node quadrilateral shell element having a cubic (translations and rotations) bending field and a linear/cubic (in-plane translation/transverse rotations) membrane field. The 410 element has three translational and three rotational degrees of freedom per node. Only geometric nonlinearities are considered, with the material modeled as linearly elastic. Arbitrarily large rotations, but small strains, are modeled using the standard nonlinear corotational procedure [16], and the Riks arc length [17] (or path parameter) technique is used to treat load and/or deflection reversals. In the case of load and deflection reversals such as will be encountered in the benchmark, dynamic analyses are required to predict the shell response if the shell is loaded only in load or deflection control. Only static analyses are considered here and are typically used to fully characterize such nonlinear problems when path-parameter techniques are used. Mesh convergence studies indicate that a 10 by 10 discretization (in the x and y directions), or 100 elements, adequately captures the load-deflection and mode evolutions for the benchmark problem [13]. Several traditional techniques for inducing bifurcation in nonlinear problems are employed and discussed (including their limitations) in Sec. II.C.

B. Incorrect-Symmetric Solution

The load vs center deflection w_c response from the STAGS analysis is presented in Fig. 3, along with the original Sabir and Lock [1] FE analysis from the book by Crisfield [5]. Both load and deflection are taken positive in the loading direction shown in Fig. 2. The limit-point solution corresponds to a symmetrically meshed STAGS model with ten elements in both the axial and circumferential directions. Line markers for the STAGS load-deflection solutions in this work represent points at which the path parameter is incremented. This rather coarse 10 by 10 element mesh represents a converged solution confirmed by a more refined 20 by 20 element mesh and accurately repeats the incorrect-symmetric response found in the literature, including the load-limit point at ~580 N (~130 lbf) and a deflection-limit point at $w_c \approx 17$ mm. Clearly, the path-parameter technique is needed for this solution, because both load and deflection reversals occur. Snap-through would occur at the load-limit point in load control, with snap-down/snap-back at the deflection-limit point in deflection control.

Spanwise mode shapes along the shell centerline are presented in Fig. 4 for several values of applied center deflection w_c. Line markers on this and subsequent numerical mode/deformation plots are at mesh grid points. The spanwise mode shapes are symmetric about the centerline and loading point. At larger values of center deflection, the shell is fully inverted (concave up, rather than the convex undeformed shape) and begins to act like a stretched membrane. This is also evident on the load-deflection behavior in Fig. 3, in which slight nonlinear stiffening is observed for center deflections greater than ~22 mm. In terms of deformations, the mode shapes in both the axial and circumferential direction are symmetric with regard to the midpoint of the shell. The benchmark solution does exhibit significant axial transverse deflection gradients, in contrast to an

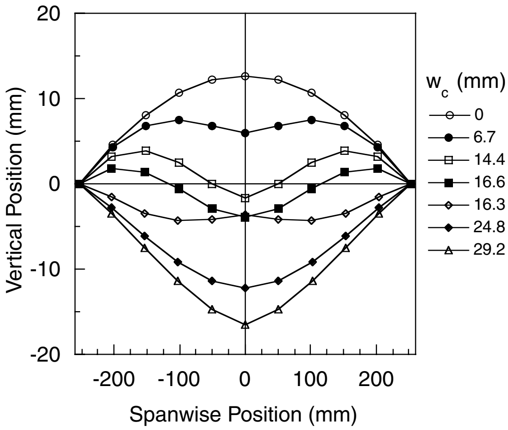

Fig. 4 FE prediction (incorrect-symmetric) of central spanwise mode shapes for the benchmark shell problem at different values of center deflection w_c.

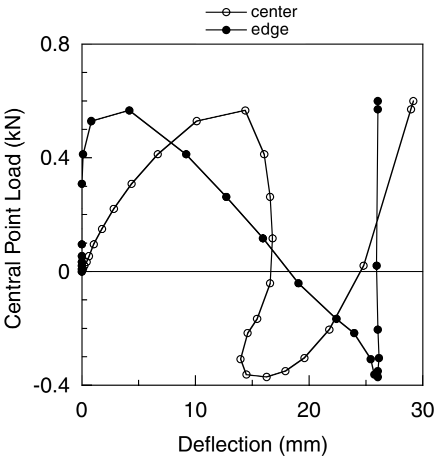

Fig. 5 FE prediction (incorrect-symmetric) of load vs center and load vs edge (midpoint of free circumferential edge) deflections.

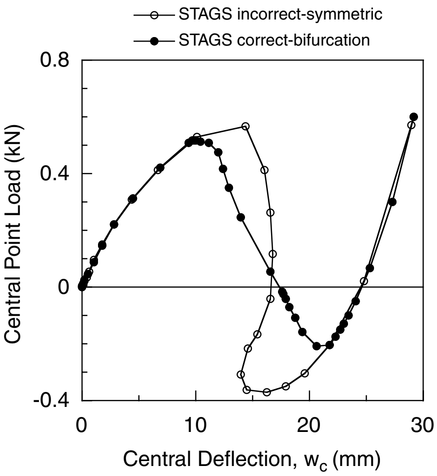

Fig. 6 FE predictions of load vs w_c for the incorrect-symmetric and correct-bifurcation solutions. Both analyses use a 10 by 10 mesh.

arch, and this is oftentimes illustrated by considering load vs center deflection and load vs edge deflection (e.g., [2]), as shown in Fig. 5. Edge deflection is the transverse deflection at the midpoint of the free edge, and so the two curves compare the response at the point of load application with the response of the free edge. The center deflection leads the edge deflection initially, as would be expected, and does so until the deflection-limit point is reached and snap-back/snap-down occurs, in which the edge snaps down (or back) ahead of the center of the shell. As the response begins to stiffen (slightly nonlinearly) beyond a center deflection of ~22 mm, the center deflection overtakes the edge deflection at ~200 N. At this point, the shell is inverted and the load applied at the center creates more deflection at that point than at the free edge and the shell is undergoing primarily membrane stretching.

The response of the benchmark shown in Figs. 3–5 contains various interesting behaviors and analysis challenges, due to the load- and deflection-limit points; it is clear why this problem makes for an interesting benchmark case. Of course, as will be shown in the next section, these solutions are incorrect because a lower-energy equilibrium path exists.

C. Correct-Bifurcation Solution

Solutions from the literature (presented in the last section as incorrect-symmetric results) make the implicit assumption that limit-point buckling will occur and thus do not consider bifurcation and the asymmetric archlike mode in Fig. 1. Most analyses in the literature make this implicit assumption by employing a quarter-model of the shell, thereby forcing symmetry of the geometry, loading, and the response. Without this implicit assumption, a full model of the shell shows that bifurcation occurs in this problem before the load-limit point of the incorrect-symmetric solution, changing the entire form of the benchmark response. Bifurcation is shown using two traditional techniques for identifying and initiating bifurcation, or a "branch switching" in FE models: both methods make use of the eigenmode(s) of the shell. First, the correct-bifurcation solution is presented and discussed, followed by a brief discussion of the issues/difficulties in identifying and initiating bifurcation in nonlinear FE problems using the benchmark as an example.

The correct-bifurcation load-deflection response is presented in Fig. 6, along with the incorrect-symmetric response for comparison. At ~510 N, the shell bifurcates onto a secondary equilibrium path associated with a dominant asymmetric mode that is comparable with that for an arch. Because bifurcation occurs at ~12% below the limit point in the incorrect-symmetric solution, this is the preferred lower-energy path. The bifurcation path is stable in load control; that is, the slope of the tangent stiffness is positive directly after the bifurcation point. In the correct-bifurcation solution, a load-limit point is reached at ~520 N, very close to the bifurcation point at ~510 N. The correct-bifurcation response does not exhibit a deflection-limit point (snap-back/snap-down), but does progress into the negative load region, reaching a minimum of ~ -205 N. The correct-bifurcation response rejoins the load-deflection path of the incorrect-symmetric response, as expected, at large values of center deflection. In this region, the response is dominated by membrane stretching (and also dominated by symmetric spanwise transverse deformations).

The qualitative difference between the incorrect-symmetric and correct-bifurcation responses of the benchmark shell is that of symmetric spanwise transverse deformation modes vs asymmetric deformation modes, respectively. As noted earlier, this behavior is qualitatively similar to shallow vs deep arches that display such behavior in one dimension. Mode-shape evolutions for the correct-bifurcation response are provided in Fig. 7 at w_c values at interesting points on the load-deflection curve in Fig. 6. The modes are symmetric through w_c values of 9.4 mm; $w_c = 9.4$ mm is very close to, but before, the bifurcation point. A slight asymmetry is noted for the mode in which $w_c = 10.0$ mm, which corresponds to the correct-bifurcation load-limit point at ~520 N. The mode shape at $w_c = 16.6$ mm is clearly dominated by an asymmetry and corresponds to the zero load-crossing point. By contrast, the last two mode shapes have evolved to nearly fully symmetric shapes. At $w_c = 21.8$ mm, at the minimum load of ~ -205 N, the shell has inverted and is fully concave upward. At increased loads, the concave shell configuration stiffens as the shell responds primarily in a membrane-stretching mode, which can be seen in the last mode shape at $w_c = 29.2$ mm (600 N). To complete the correct-bifurcation benchmark presentation, the load vs center and edge deflections are presented in Fig. 8 for comparison with Fig. 5.

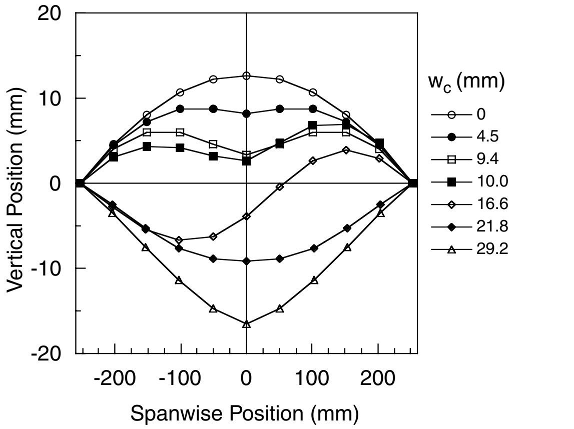

Fig. 7 FE prediction (correct-bifurcation) of central spanwise mode shapes for the benchmark shell problem at different values of center deflection w_c.

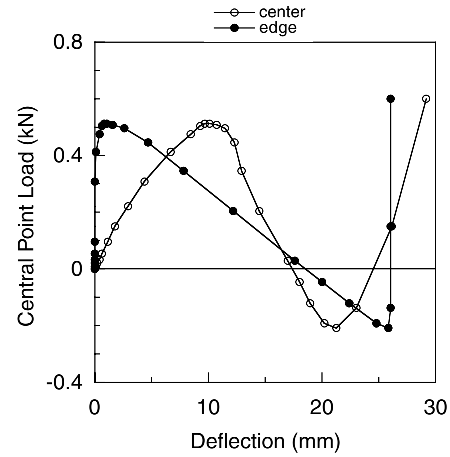

Fig. 8 FE prediction (correct-bifurcation) of load vs center and load vs edge (midpoint of free circumferential edge) deflections for the benchmark.

The correct-bifurcation solution has been generated using traditional techniques available in STAGS for identifying and inducing bifurcation. Results from two such techniques will be briefly presented using the benchmark problem to illustrate typical issues involved in such analyses. (For example, path-parameter techniques can miss, or "step over," bifurcation points during a nonlinear analysis [18,19]; this is the case in the STAGS results for the incorrect-symmetric response in Fig. 3.) Lack of identifying the bifurcation point in the benchmark is clearly an issue in the incorrect-symmetric responses that have persisted in the literature. The two bifurcation techniques are 1) introducing a geometric imperfection in the shell geometry in the form of the first linear eigenmode and 2) using the "equivalence transform" (ET) technique, which uses the eigenmode of the tangent stiffness matrix as an assumed incremental displacement solution near the bifurcation point. The second is considered more sophisticated because it uses the nonlinear bifurcation mode and introduces the assumed solution only near the bifurcation point. Identification of the bifurcation point involves monitoring the determinant of the tangent stiffness matrix [20] along the nonlinear load-deflection path, and through successive iterations, the bifurcation point can be pinpointed [14]. These two techniques are specific manifestations of a broad class of techniques that involve introducing imperfections into the FE analysis; they are further specific in that they employ several subjective choices made in the course of the analysis (e.g., use of only the first eigenmode instead of multiple modes). Finally, it is important to point out the nonlinear vs linear bifurcation-point calculations. The incorrect-symmetric models in Sec. II.B predict the linear bifurcation point to occur at 661 N, *above* the incorrect-symmetric load-limit point at 580 N. The nonlinear bifurcation point at 510 N from the correct-bifurcation solution in this section is *below* the incorrect-symmetric load-limit point. Relying solely on a linear bifurcation analysis results in the shell response taking on the nature of the incorrect-symmetric solution.

The loading response for the benchmark using the first technique (geometric imperfections) is shown in Fig. 9 for three different imperfection amplitudes. The largest translational component of the first linear eigenmode (illustrated in the Fig. 9 inset), scaled to the shell thickness, is incorporated as an imperfection in the initial shape of the shell. The smallest imperfection amplitude ($0.1\%t$) completely misses bifurcation, and the incorrect-symmetric response is computed (compare with Fig. 3). The imperfection is too small to perturb the problem and generate the correct-bifurcation solution. The larger imperfection amplitudes are sufficient to induce bifurcation, but illustrate that a convergence exercise must be undertaken wherein the imperfection amplitude must be large enough to induce bifurcation, but small enough to not change the problem (and thus the solution); the $10\%t$ solution in Fig. 9 is clearly different from the correct-bifurcation solution in Fig. 6. As revealed in Fig. 10, the second (ET) technique can also miss the bifurcation point if the assumed solution magnitude near the bifurcation point is

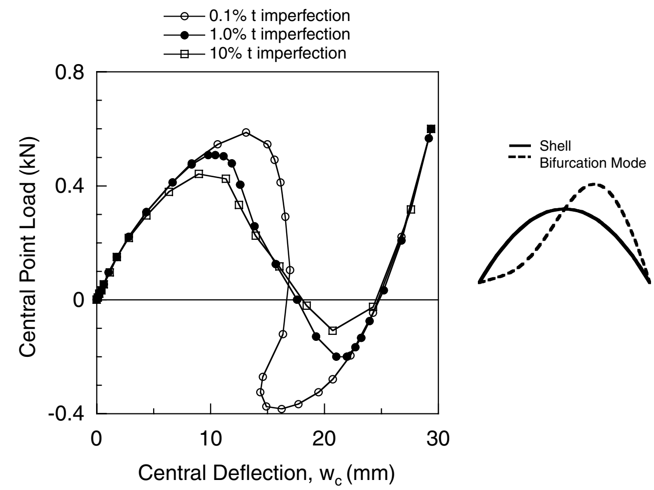

Fig. 9 Predicted load vs w_c for the benchmark problem using geometric imperfections; the inset illustrates shape of first linear eigenmode.

too small ($1.0\%t$), but adequately captures the postbifurcation response even for relatively large assumed amplitudes ($10\%t$). Bifurcation is induced at 509 N using the first eigenmode from the tangent stiffness matrix at this load. The correct-bifurcation solutions presented in Sec. II.C all correspond to the $2.0\%t$ ET technique.

Although various techniques exist for inducing bifurcation, it is important to note that the user must initiate the bifurcation analysis. It is also important to note that even in the case of the very simple geometry of the benchmark problem, various subjective choices need to be made by the user. In an effort to reduce the number of subjective choices and to improve the efficiency (from the users' time perspective) of such analyses, an alternative technique for identifying and inducing bifurcation in nonlinear FE problems was developed, called the *asymmetric meshing technique* (AMT) [13]. All solutions in this paper were confirmed using traditional bifurcation techniques. The load-deflection and mode-shape evolutions for the benchmark using the AMT are presented in Fig. 11 [13].

Although the AMT is not the focus of this paper, it is discussed here briefly for two reasons:

1) The correct-bifurcation solution was first discovered using the AMT and only later confirmed using traditional techniques (including the two presented here).

2) It was used to efficiently evaluate the response of a large number (~75) of experimental composite shell specimens, including the shell in the next section.

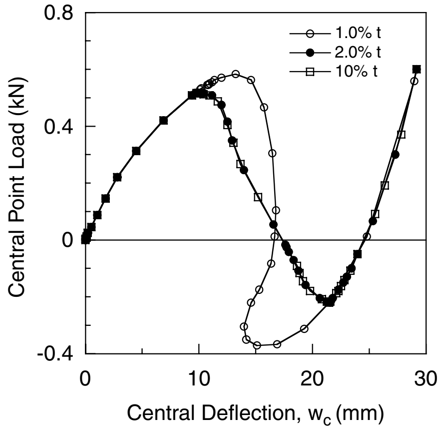

Fig. 10 Predicted load vs w_c for the benchmark using the equivalence transform technique.

The AMT is considered to be efficient because the full solution to a given cylindrical shell problem is developed in a single FE run, whereas the traditional techniques require 5–10 runs that necessarily include subjective choices to be made by the user after interpreting the prior run's results; that is, the user must adjust a geometric imperfection amplitude so that it is large enough to cause bifurcation, but not so large as to change the problem (see Figs. 9 and 10 and the surrounding discussion). The principle of introducing either geometric or loading imperfections, more accurately termed asymmetries, to induce bifurcation in the FE analysis is exploited by introducing an asymmetry into the mesh. The mesh asymmetry provides a numerical asymmetry, just as for the traditional techniques, that allows nonlinear algorithms in the FE solver to consider asymmetric solutions. In the case of the AMT, the numerical effect of the asymmetric mesh is similar to that caused by a geometric imperfection, not a loading asymmetry, because the stiffness matrix $\mathbf{K}$ is affected rather than the loading vector $\mathbf{F}$ in $\mathbf{F} = \mathbf{K}\mathbf{x}$. A discussion of the tangent stiffness matrix in relation to buckling of structures in FE formulations can be found in Chapter 10 of [10]. Additional exploration and quantification of the AMT's advantages and identification of any issues should be pursued, but is left as future work.

III. Composite Shell as a Potential New Benchmark Problem

As indicated earlier, no experimental data exist for the benchmark problem, and therefore no verification of solutions can take place. Such experimental data would have illuminated the correct-bifurcation behavior of the benchmark immediately. It is suggested that a composite shell-buckling problem, in which both load-deflection and mode-shape data exist [21], makes a better benchmark (at least until experimental data are generated) than the Sabir and Lock [1] benchmark. The problem is very similar to the prior benchmark, except that the shell is a graphite-epoxy (AS4/3501-6) laminated $[\pm 45/0]_s$ composite shell, rather than isotropic. The boundary conditions are the same as the benchmark problem: the shell is experimentally loaded in deflection control with a 12.7-mm ($\frac{1}{2}$-in.) hemispherical-diameter steel indenter that approximates a point load, and the shell is modeled as a perfect shell (not including any measured imperfections), with dimensions and ply properties given in Table 1. Results presented here for the composite shell are obtained with the AMT, but were validated by traditional bifurcation techniques. The mesh used is converged and consists of 10 elements in the axial direction and an asymmetric mesh of 11 elements in the spanwise direction. Additional details on the experiments and analysis are available [13,22,23].

Table 1 Composite shell data

Shell parameter	Dimension
Side length L	304.8 mm (12 in.)
Radius R	881 mm (34.7 in.)
Span S	304.8 mm (12 in.)
Thickness t	0.804 mm
Height h	13.3 mm (0.52 in.)
Ply E_{11}	142 GPa
Ply E_{22}	9.81 GPa
Ply G_{12}	6.00 GPa
Ply ν_{12}	0.3
Ply thickness	0.134 mm

The composite shell undergoes bifurcation buckling in much the same way as the Sabir and Lock [1] benchmark. A (symmetric mode only) limit-point solution can be calculated for this shell, but the experimental data clearly show bifurcation into an asymmetric mode, as seen in Figs. 12 and 13. Excellent agreement between the nonlinear STAGS FE analysis (when bifurcation is considered) and the experimental data over large values of normalized center deflection w_c/t support this shell problem as a potential new benchmark. Note that only positive load could be applied to the shell, because after reaching zero load (after bifurcation), the experimental shell snapped down into a concave up configuration and the indenter lost contact with the shell. Finally, axial spanwise mode shapes are presented in Fig. 14 to complete the potential benchmark. Note that an initial imperfection is evident in the undeformed experimental axial mode shape in Fig. 14, because it is not straight; such imperfections are unmodeled, but do not strongly affect either the load-deflection or spanwise mode shapes. This imperfection is likely due to residual thermal-strain-induced bending from curing/manufacturing [23]. Note also in Fig. 14 that the free shell edges do not deform symmetrically as in the isotropic benchmark (this is most clear in the predicted results in Fig. 14); this is attributed to bending–

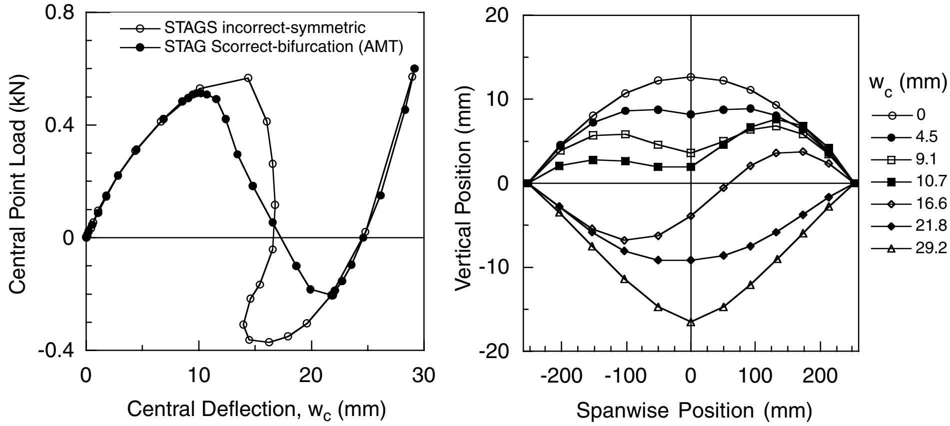

Fig. 11 FE predictions of load vs w_c and central spanwise mode shapes for the benchmark considering bifurcation using the alternative AMT.

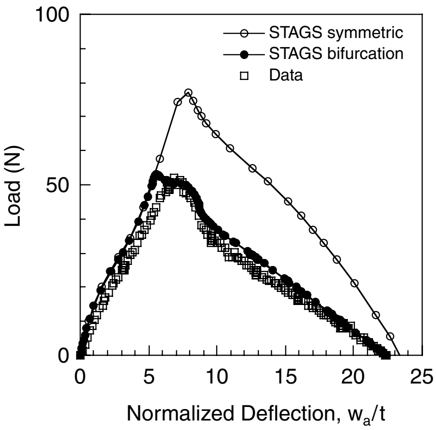

Fig. 12 Load vs w_c/t for the composite shell including predicted (STAGS FE) and experimental results.

twisting coupling in the laminate and is discussed in detail for a similar composite shell [22].

IV. Conclusions

The benchmark isotropic shell problem dating back to Sabir and Lock [1] in 1972 is shown to be incorrect and the correct-bifurcation solution is presented and discussed. The correct-bifurcation solution involves branching from the nonlinear equilibrium path to an asymmetric spanwise mode, and the overall behavior (including the mode shape) is similar to the behavior of deep arches. Because no experimental data exist to verify any of the analyses for the benchmark shell problem, a composite shell problem that is very similar to the Sabir and Lock [1] benchmark is suggested as a possible alternative benchmark. The FE predictions for the composite shell are in good agreement with experimental load-deflection and mode-shape evolutions when bifurcation is considered.

It should be noted that the Sabir and Lock [1] benchmark (incorrect-symmetric) is useful for the specific purpose of testing the ability of FE codes and associated numerical procedures to traverse load/deflection-reversal points in nonlinear analyses. It is useful in such cases because the limit-point solution is an equilibrium path; however, it is an unstable equilibrium path (and therefore the incorrect solution) and will not be realized in an experiment with such a structure.

Two suggestions are made for further work aimed at improving the current state of the isotropic benchmark: First, experimental data are needed for the benchmark problem to make it a truly useful benchmark. Both load-deflection and mode-shape data should be acquired, similar to that for the composite shell problem. However, in addition, detailed measurements of initial imperfections and loading anomalies suitable for incorporation in high-fidelity numerical analyses should also be acquired, as suggested by others [24]. Experimental data would have illuminated the incorrect-symmetric benchmark solution and not have allowed it to persist in the literature for over three decades. Second, there may be more utility in creating a shell configuration that varies slightly from the original benchmark to test and acquire data. The original benchmark is of very low modulus (likely, plastic), and data will likely suffer from contributions associated with local deformations at the point load; further, the correct-bifurcation solution contains an interesting but small (~10 N of applied load) region of stable postbifurcation behavior that will be very difficult to discern even in the best experiment. Creation of a small family of isotropic benchmarks that display the characteristics of both the correct-bifurcation response and the incorrect-symmetric response (especially snap-back/snap-

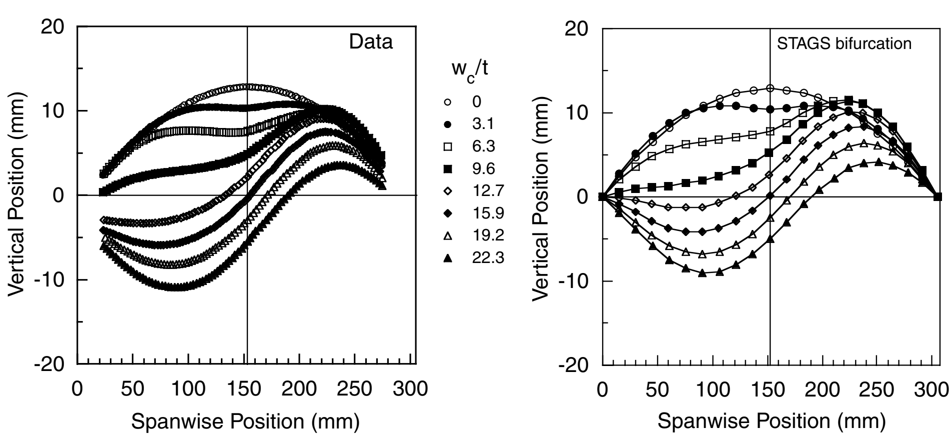

Fig. 13 Experimental (left) and predicted (right) central spanwise mode shapes for the composite shell at different values of w_c/t.

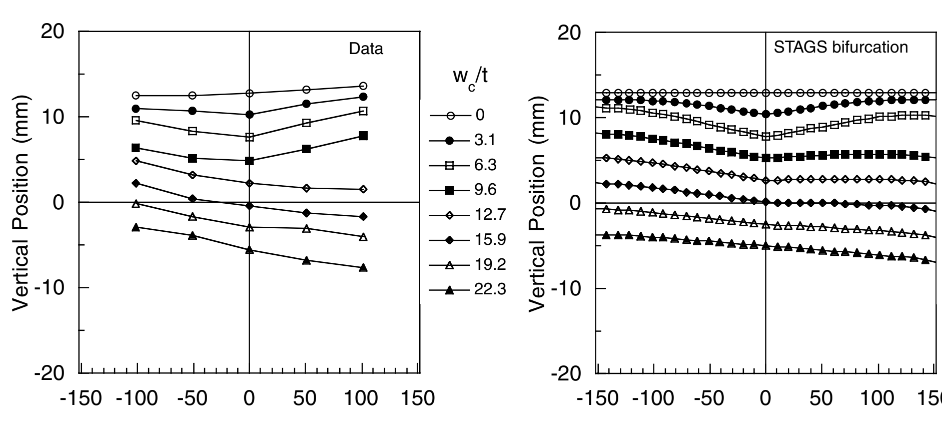

Fig. 14 Experimental (left) and predicted (right) central axial mode shapes for the composite shell at different values of w_c/t.

down behavior) is suggested as an alternative. Proper variation of the basic shell parameters, such as thickness and height, should easily allow this. Finally, the asymmetric meshing technique (AMT) for identifying and traversing bifurcation points in nonlinear FE formulations was shown to be efficient and accurate for this class of shell-buckling problems. The technique should be further explored to discern limitations and quantify advantages.

Acknowledgments

This work was sponsored by the Federal Aviation Administration under research grant 94-G-037 and NASA Langley Research Center under NASA grant NAG-1-991. The author gratefully acknowledges the help of Paul A. Lagace (Massachusetts Institute of Technology) and John W. Hutchinson (Harvard University), who advised during this work; Robert M. Jones (Virginia Polytechnic Institute and State University) for encouraging the writing of this manuscript; and Mark A. Tudela (U.S. Air Force Research Laboratory, Wright–Patterson Air Force Base) for the use of experimental data. The author wishes to acknowledge the helpful advice of the late James (Jim) H. Starnes, Jr. of NASA.

References

[1] Sabir, A. B., and Lock, A. C., "The Application of Finite Elements to the Large Deflection Geometrically Non-Linear Behaviour of Cylindrical Shells," *Variational Methods in Engineering*, edited by C. A. Brebbia and H. Tottenham, Southampton Univ. Press, Surrey, England, U.K., Vol. 2, 1973, pp. 7/66–7/75.

[2] Kim, K. D., Liu, G. Z., and Han, S. C., "A Resultant 8-Node Solid-Shell Element for Geometrically Nonlinear Analysis," *Computational Mechanics*, Vol. 35, No. 5, Apr. 2005, pp. 315–331. doi:10.1007/s00466-004-0606-9

[3] Lee, S. J., and Hinton, E., "Dangers Inherited in Shells Optimized with Linear Assumptions," *Computers and Structures*, Vol. 78, Nos. 1–3, Nov. 2000, pp. 473–486. doi:10.1016/S0045-7949(00)00083-3

[4] Palazotto, A. N., Chien, L. S., and Taylor, W. W., "Stability Characteristics of Laminated Cylindrical Panels Under Transverse Loading," *AIAA Journal*, Vol. 30, No. 6, June 1992, pp. 1649–1653.

[5] Crisfield, M. A., *Non-Linear Finite Element Analysis of Solids and Structures*, Wiley, New York, Vol. 1, 1991, pp. 268–269.

[6] Simo, J. C., Fox, D. D., and Rifai, M. S., "Geometrically Exact Stress Resultant Shell Models: Formulation and Computational Aspects of the Nonlinear Theory," *Analytical and Computational Models of Shells, Winter Annual Meeting of the ASME*, edited by A. K. Noor, T. Belytschko and J. C. Simo, American Society of Mechanical Engineers, New York, 1989, pp. 161–190.

[7] Bergan, P. G., Horrigmoe, G., Krakeland, B., and Soreide, T. H., "Solution Techniques for Non-Linear Finite Element Problems," *International Journal for Numerical Methods in Engineering*, Vol. 12, No. 11, 1978, pp. 1677–1696. doi:10.1002/nme.1620121106

[8] "1.1.6 Snap-Through of a Shallow, Cylindrical Roof Under a Point Load," *ABAQUS Benchmarks Manual* [online manual], Ver. 6.6, http://lcadm.rrzn.uni-hannover.de:2080/v6.6/books/bmk/default.htm [retrieved 7 Dec. 2007].

[9] Bazant, Z. P., and Cedolin, L., *Stability of Structures*, Oxford Univ. Press, Oxford, 1991, Sec. 4.8.

[10] Timoshenko, S. P., and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw–Hill, Singapore, 1963, Sec. 10.3.

[11] Fung, Y. C., and Kaplan, A., "Buckling of Low Arches or Curved Beams of Small Curvature," NACA TN 2840, Nov. 1952.

[12] Johnson, E. R., Hyer, M. W., and Carper, D. M., "Response of Composite Material Shallow Arch to Concentrated Load," *AIAA Journal*, Vol. 23, No. 6, June 1986, pp. 529–536.

[13] Wardle, B. L., "Buckling and Damage Resistance of Transversely Loaded Composite Shells," Ph.D. Thesis, Department of Aeronautics and Astronautics, Massachusetts Inst. of Technology, Cambridge, MA, June, 1998; also Technology Lab. for Advanced Composites, Rept. 98-7, Cambridge, MA, 1998.

[14] Brogan, F. A., Rankin, C. C., Cabiness, H. D., and Loden, W. A., "STAGS User Manual: Version 2.3," Lockheed Martin Missiles and Space Co., Rept. PO32594, Sunnyvale, CA, July 1996.

[15] Rankin, C. C., and Brogan, F., "The Computational Structural Mechanics Testbed Structural Element Processor ES5: STAGS Shell Element," NASA CR-4358, 1991.

[16] Rankin, C. C., and Brogan, F. A., "An Element-Independent Corotational Procedure for the Treatment of Large Rotations," *Collapse Analysis of Structures*, American Society of Mechanical Engineers, New York, 1984, pp. 85–100.

[17] Riks, E., "Some Computational Aspects of the Stability Analysis of Nonlinear Structures," *Computer Methods in Applied Mechanics and Engineering*, Vol. 47, 1984, pp. 219–259. doi:10.1016/0045-7825(84)90078-1

[18] Riks, E., "Bifurcation and Stability, A Numerical Approach," *Innovative Methods for Nonlinear Problems*, edited by W. K. Liu, T. Belytschkoand, and K. C. Parks, Pineridge Press, Swansea, Wales, 1984, pp. 313–344.

[19] Riks, E., Brogan, F. A., and Rankin, C. C., "Numerical Aspects of Shell Stability Analysis," *Computational Mechanics of Nonlinear Response of Shells*, Spring–Verlag, New York, 1990, pp. 125–151.

[20] Bazant, Z. P., and Cedolin, L., *Stability of Structures*, Oxford Univ. Press, Oxford, 1991, Sec. 4.5.

[21] Tudela, M. A., Lagace, P. A., and Wardle, B. L., "Buckling Response of Transversely Loaded Composite Shells, Part 1: Experiments," *AIAA Journal*, Vol. 42, No. 7, July 2004, pp. 1457–1464.

[22] Wardle, B. L., Lagace, P. A., and Tudela, M. A., "Buckling Response of Transversely Loaded Composite Shells, Part 2: Numerical Analysis," *AIAA Journal*, Vol. 42, No. 7, July 2004, pp. 1465–1473.

[23] Tudela, M. A., "Structural Response and Damage Development of Cylindrical Composite Panels," S.M. Thesis, Dept. of Aeronautics and Astronautics, Massachusetts Inst. Technology, Cambridge, MA, Sept. 1996; also Technology Lab. for Advanced Composites, Rept. 96-11, Cambridge, MA, 1996.

[24] Singer, J., Abramovich, H., and Weller, T., "The Prerequisites for an Advanced Design Methodology in Shells Prone to Buckling," 42nd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference and Exhibit, AIAA Paper 2001-1396, 2001.

A. Roy
Associate Editor